VERIFICATION.—In accordance with the conditions, the rate of dividend of Doe's estate is found to be 20.1293 per cent of the liabilities. And the rate of dividend of Roe's estate is found to be 19.4399 per cent of the liabilities. These rates can be verified as follows:

| | | |
|---|---|---|
| John Doe's estate—direct liabilities, | | \$33,425.61 |
| Endorsements for Richard Roe | \$34,949.16 | 28,155.08 |
| Less dividend from " " | 6,794.08 | |
| Total liabilities, | | \$61,580.69 |

Total dividends \$12,395.76, equal to 20.1293 per cent of liabilities.

| | | |
|---|---|---|
| Richard Roe's estate—direct liabilities, | | \$46,212.00 |
| Endorsements for John Doe | \$9,500.00 | 7,587.72 |
| Less dividend from " " | 1,912.28 | |
| Total liabilities, | | \$53,799.72 |

Total dividends, \$10,458.78, equal to 19.4399 per cent of total liabilities.

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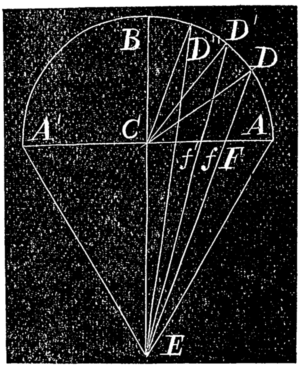
## *APPROXIMATE MULTISECTION OF AN ANGLE AND HINTS FOR REDUCING THE UNAVOIDABLE ERROR TO THE SMALLEST AMOUNT.*

BY CHAS. H. KUMMELL, DETROIT, MICHIGAN.

THE method of Query, page 96, is applicable also for multisection of angles or for dividing angles in a given ratio, approximately.

For example, if $BCD$ shall be trisected, draw $ACA'$ perpendicular to $BC$ and describe, with any radius $CA$, the semicircle $ADBA'$; make $AE = A'E = AA'$, join $ED$ intersecting $AA'$ at $F$; trisect $CF$ at $f$ and $f'$ and draw $EfD'$ and $Ef'D''$, then the arc $BD$ will be approximately trisected in $D'$ and $D''$.




The answer to the query by Mr. E. B. Seitz at page 125, 126 is quite sufficient to prove this construction to be approximately true; yet for my purpose I shall present a different treatment.

The lines $Ef$ and $Ef'$ will intersect the circle $ABA'$ at points $D'$ and $D''$ which are more or less distant from the true points required; they may also be on the right or on the left of the true points. Let $CA = CB = 1$; $BCD'' = \varphi$; $BED'' = \psi$; $Cf' = x$. Let $\varphi_0$ be the true angle, then

$$\Delta = \varphi_0 - \varphi \tag{1}$$

is the correction to the constructed angle to obtain the true angle.

By construction we have

$$x = \frac{2\varphi_0}{\pi}. \tag{2}$$

In the triangle $ECD''$ we have $CD'' = 1$; $CE = \sqrt{3}$; $ECD'' = \pi - \varphi$. Therefore

$$\tan\psi = \frac{\sin\varphi}{\sqrt{3}+\cos\varphi}. \tag{3}$$

Now
$$x = \sqrt{3}\,.\tan\psi = \frac{\sqrt{3}\,.\sin\varphi}{\sqrt{3}+\cos\varphi}; \tag{4}$$

$\therefore$ by (2)
$$\varphi_0 = \frac{\pi}{2}.\frac{\sqrt{3}\,.\sin\varphi}{\sqrt{3}+\cos\varphi}, \tag{5}$$

and by (1)
$$\Delta = \frac{\pi}{2}.\frac{\sqrt{3}\,.\sin\varphi}{\sqrt{3}+\cos\varphi} - \varphi. \tag{6}$$

This equation gives $\Delta = 0$ if $\varphi = 0$, $\varphi = \frac{1}{6}\pi$, or $\varphi = \frac{1}{2}\pi$.

To find the maximum and minimum values of $\Delta$ we place

$$\frac{d\Delta}{d\varphi} = \tfrac{1}{2}\pi\sqrt{3}\,\frac{1+\sqrt{3}\,.\cos\varphi}{(\sqrt{3}+\cos\varphi)^2} - 1 = 0,$$

or
$$\cos^2\varphi - 2(\tfrac{3}{4}\pi - \sqrt{3})\cos\varphi = \tfrac{1}{2}\pi\sqrt{3} - 3,$$

whence
$$\cos\varphi = \tfrac{3}{4}\pi - \sqrt{3} \pm \sqrt{(\tfrac{9}{16}\pi^2 - \pi\sqrt{3})}.$$

From this we have, for the upper sign,

$$\varphi_+ = 17°01'21''; \;\therefore\; \Delta_+ = -0.000817 = -02'48'' \text{ (minimum)},$$

and for the lower sign,

$$\varphi_- = 73°01'01''; \;\therefore\; \Delta_- = +0.011327 = +38'56'' \text{ (maximum)}.$$

We learn from this that the constructed angle is slightly too great in the interval $\varphi = 0$ to $\varphi = \frac{1}{6}\pi$, the greatest error $2'48''$ occurring at about $17°$ or near the middle of the interval. Also the constructed angle is by a greater amount too small in the interval $\varphi = \frac{1}{6}\pi$ to $\varphi = \frac{1}{2}\pi$, the greatest error $38'56''$ occurring at about $73°$ or at nearly $\frac{1}{4}$ from the end of the interval.

Having multisected an angle by this method, the parts will not be strictly equal, but we can decide now which is freest of error. The first choice should be that part extending nearly at equal distances from the minimum point; the second best would be a part $\frac{1}{4}$ of which extends beyond the maximum point, and the third best, any part about the minimum interval.

In any case the error of draughting may or may not combine favorably with these theoretical errors, but this we cannot control and we have to leave these accidental errors out of consideration.

It should be observed that if $\varphi$ is negative the errors are numerically equal but of opposite signs to those in the positive quadrant.

NOTE.—Equation (2) is strictly true, however, only if $\varphi = \frac{1}{6}\pi$ or $= \frac{1}{2}\pi$, and for the approximate construction of a polygon. For any angle $\varphi$ to be multisected we have, if $\varphi_1$ is one part reckoned from $B$, strictly

$$\frac{x}{x_1} = \frac{\varphi}{\varphi_1} \text{ where } x = \frac{r\sqrt{3}\,.\sin\varphi}{\sqrt{3}+\cos\varphi}, \text{ and } x_1 = \frac{r\sqrt{3}.\sin(\varphi_1 - \Delta)}{\sqrt{3}+\cos(\varphi_1-\Delta)}$$

and $\Delta$ is the correction to be applied to the angle $\varphi_1 - \Delta$, which the construction gives. We have then the equation:

$$0 = \varphi(\sqrt{3}+\cos\varphi)\sin(\varphi_1-\Delta) - \varphi_1\sin\varphi[\sqrt{3}+\cos(\varphi_1-\Delta)],$$

and the maximum or minimum condition is

$$0 = \varphi(\sqrt{3}+\cos\varphi)\cos(\varphi_1-\Delta) - \sin\varphi[\sqrt{3}+\cos(\varphi_1-\Delta)] + \varphi_1\sin\varphi \times\sin(\varphi_1-\Delta).$$

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## *CORRESPONDENCE.*

EDITOR ANALYST:

Since reading your note on page 150 of the ANALYST, I have made some vain essays at a demonstration of the proposition stated in Problem 211. Though I failed in my main object, some of the results I attained in regard to triangular numbers seem to me of some interest, and may appear to you of sufficient interest to warrant their insertion in your Journal.

JOHN MACNIE.

Let $N = \frac{1}{2}(n^2+n) = t_n$ be the $n$th triangular number, then

$$8N+1 = (2n+1)^2.$$

Hence, in order that an integer $N$, be a triangular number, it is necessary and sufficient that $8N+1$ be a square number. Also, any odd square number is of the form $8t_n+1$.

Ex. 1. Let $N = 55$; $8N+1 = 441 = 21^2$. $\therefore 55 = t_{10}$.

In the same way we find that for $N$ to be the sum of two triangular numbers, $t_n$ and $t_p$, it is necessary and sufficient that $8N+2 = (2n+1)^2 + (2p+1)^2$, i.e., be the sum of two odd squares.

Ex. 2. Let $N = 93$; $8N+2 = 746 = 25^2+11^2$. $\therefore 93 = t_{12} + t_5$.